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Combinatorially Homogeneous Graphs

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Gardiner classified ultrahomogeneous graphs and posed the problem of defining "combinatorial homogeneity." Later, Ronse proved that homogeneous graphs are ultrahomogeneous by classifying such graphs. In this paper, we give a direct proof that (suitably defined) combinatorially homogeneous graphs are ultrahomogeneous. Also, we classify combinatorially C-homogeneous graphs.

In this paper, we study finite, simple, undirected graphs, and follow notations used in [1] or [2]. For a graph Γ, $V\Gamma$ is the set of vertices of Γ, and for $x \in V\Gamma$,

$$\Gamma_i(x) = \{y \in V\Gamma \mid d(x, y) = i\},$$

where d is the distance function. We also write $\Gamma(x)$ in place of $\Gamma_1(x)$. For a subset X of $V\Gamma$, the corresponding vertex-subgraph is denoted by $\langle X\rangle$. For a graph Γ, $d(\Gamma)$ denotes the diameter of Γ, $L(\Gamma)$ the line graph of Γ, Γ^c the complement graph of Γ, and $t \cdot \Gamma$ the disjoint union of t copies of Γ. C_n is the circuit graph on n vertices; K_{1+k} is the complete graph of valency k; $K_{r_1,r_2,\ldots,r_m}$ is the complete m-partite graph having parts of size $r_1, r_2, \ldots, r_m$; $K_{t;r} = (t \cdot K_r)^c$ is the regular complete t-partite graph; Q_k is the k-dimensional cube; $\square_k$ is the graph obtained by identifying antipodal vertices of Q_k.

A graph Γ is said to be *homogeneous* if, whenever vertex-subgraphs $\langle X\rangle$ and $\langle X'\rangle$ are isomorphic, then there is an automorphism of Γ taking X to X'. Γ is *ultrahomogeneous* if every isomorphism of $\langle X\rangle$ onto $\langle X'\rangle$ extends to an automorphism of Γ. In [2], Gardiner completed Sheehan's classification [6] of ultrahomogeneous graphs and posed the problem of defining "combinatorial homogeneity." We propose the following:

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DEFINITION. For a subset $X \subset V\Gamma$, we define

$$\Gamma(X) = \bigcap_{x \in X} \Gamma(x),$$

that is, the set of vertices adjacent to all the vertices in X. A graph Γ is *combinatorially homogeneous*, or simply K-homogeneous, if $|\Gamma(X)| = |\Gamma(X')|$ for any isomorphic vertex-subgraphs $\langle X \rangle$ and $\langle X' \rangle$.

It is obvious that ultrahomogeneous graphs are homogeneous and that homogeneous graphs are K-homogeneous. Conversely, we shall show that K-homogeneous graphs are ultrahomogeneous. This implies the equivalence of homogeneity and ultrahomogeneity, first proved by Ronse [5].

THEOREM 1. *Let Γ be a K-homogeneous graph. Then Γ is ultrahomogeneous and is isomorphic to one of the following graphs*:

(a) *a disjoint union of isomorphic complete graphs*;

(b) *a regular complete t-partite graph $K_{t;r}$, $t, r \geqslant 2$*;

(c) C_5;

(d) $L(K_{3,3})$.

Proof. Suppose two vertex-subgraphs, $\langle X \rangle$ and $\langle Y \rangle$, are isomorphic, and let σ be any isomorphism of $\langle X \rangle$ onto $\langle Y \rangle$. We shall show that σ extends to an automorphism of Γ by induction on $|V\Gamma - X|$. We have nothing to prove when $X = V\Gamma$, so we may assume that $X \neq V\Gamma$. Choose a vertex $x \in V\Gamma - X$ so as to maximize $|\Gamma(x) \cap X|$, and set $X' = \Gamma(x) \cap X$ and $Y' = (X')^\sigma$. Then $\sigma|_{X'}$ is an isomorphism of $\langle X' \rangle$ onto $\langle Y' \rangle$, and we have

$$|\Gamma(X')| = |\Gamma(Y')|$$

by the definition of K-homogeneity. On the other hand,

$$|\Gamma(Y') \cap Y| = |(\Gamma(X') \cap X)^\sigma| = |\Gamma(X') \cap X|.$$

Hence we have

$$|\Gamma(Y') \cap (V\Gamma - Y)| = |\Gamma(X') \cap (V\Gamma - X)| \geqslant 1,$$

and there exists an element $y \in \Gamma(Y') \cap (V\Gamma - Y)$. It is obvious that $Y' \subset \Gamma(y) \cap Y$ by the choice of y, while the maximality of X' implies that $Y' = \Gamma(y) \cap Y$. Now, define a map τ from $X \cup \{x\}$ onto $Y \cup \{y\}$ by $\tau|_X = \sigma$ and $x^\tau = y$. Then τ is an isomorphism of $\langle X \cup \{x\} \rangle$ onto $\langle Y \cup \{y\} \rangle$. By induction, τ extends to an automorphism of Γ, which is also an extension of σ. We have proved that Γ is ultrahomogeneous, and the classification follows from Gardiner's classification [2] of ultrahomogeneous graphs.

A graph Γ is *C-ultrahomogeneous* (ultrahomogeneous for connected subgraphs) if every isomorphism of connected vertex-subgraphs extends to an automorphism of Γ. Γ is *C-homogeneous* if, whenever vertex-subgraphs $\langle X \rangle$ and $\langle X' \rangle$ are connected and isomorphic, then there is an automorphism of Γ taking X to X'.

Gardiner [4] classified C-ultrahomogeneous graphs by classifying locally C-homogeneous graphs. The main purpose of this paper is to show that C-homogeneous graphs are C-ultrahomogeneous by classifying (suitably defined) combinatorially C-homogeneous graphs.

DEFINITION. A graph Γ is *combinatorially C-homogeneous*, or simply KC-homogeneous, if whenever two vertex-subgraphs $\langle X \rangle$ and $\langle X' \rangle$ are connected and isomorphic, then for any subset Y of X there is an isomorphism σ of $\langle X \rangle$ onto $\langle X' \rangle$ such that $|\Gamma(Y)| = |\Gamma(Y^\sigma)|$.

THEOREM 2. *A connected KC-homogeneous graph is either*

(a) *a complete graph* K_r, $r \geqslant 1$,
(b) *a circuit graph* C_n, $n \geqslant 5$,
(c) *a regular complete t-partite graph* $K_{t;r}$, $t, r \geqslant 2$,
(d) $L(K_{k,k})$, $k \geqslant 3$,
(e) $L(K_{2,k+1})^c$, $k \geqslant 3$,
(f) *Petersen's graph* $L(K_5)^c$, *or*
(g) $\square_5$.

A disconnected graph is KC-homogeneous if and only if its connected components are isomorphic KC-homogeneous graphs.

We remark that all the graphs listed in the above theorem are C-ultrahomogeneous. As an example, we shall prove the following:

PROPOSITION 3. $L(K_{k,k})$ *is C-ultrahomogeneous.*

Proof. Let $\langle X \rangle$ and $\langle Y \rangle$ be connected subgraphs of $\Gamma = L(K_{k,k})$ and σ an isomorphism of $\langle X \rangle$ onto $\langle Y \rangle$. Note that for any automorphisms σ_1 and σ_2 of Γ, $\sigma_1^{-1}\sigma\sigma_2$ is an isomorphism of $\langle X^{\sigma_1} \rangle$ onto $\langle Y^{\sigma_2} \rangle$, and that σ has an extension if and only if $\sigma_1^{-1}\sigma\sigma_2$ has an extension. We may identify $V\Gamma$ with the set of all ordered pairs of k symbols, that is,

$$V\Gamma = \{(i, j) \mid 1 \leqslant i, j \leqslant k\},$$

and (i, j) and (i', j') are adjacent if $i = i'$ or $j = j'$. If $\langle \Gamma(x) \cap X \rangle$ is complete for every $x \in X$, then X itself is complete, and it is obvious that σ has an extension in this case. Suppose $\langle \Gamma(x) \cap X \rangle$ is not complete for some $x \in X$.

By the above remark, we may assume that $x = (1, 1)$, $x^\sigma = x$ and $y^\sigma = y$ for every $y \in \Gamma(x) \cap X$. Let

$$I = \{i \mid 2 \leqslant i \leqslant k \text{ and } (i, 1) \in X\},$$
$$J = \{j \mid 2 \leqslant j \leqslant k \text{ and } (1, j) \in X\},$$
$$Z = \{(i, j) \mid 2 \leqslant i, j \leqslant k\},$$
$$X' = (X \cap Z) \cup (I \times J),$$

and

$$Y' = (Y \cap Z) \cup (I \times J).$$

We remark that $I \times J \neq \phi$ from the choice of x and that $(i, j) \in X \cap (I \times J)$ if and only if $(i, j) \in Y \cap (I \times J)$. Therefore $\langle X' \rangle$ and $\langle Y' \rangle$ are connected subgraphs of $\langle Z \rangle \cong L(K_{k-1,k-1})$, and σ induces an isomorphism σ' of $\langle X' \rangle$ onto $\langle Y' \rangle$. By induction, σ' extends to an automorphism τ of $\langle Z \rangle$. If $|X'| \leqslant 1$, the proposition is trivially true. Therefore, we may assume that $|X'| > 1$. Then there is an element $y = (i, j) \in I \times J$ such that $\Gamma(y) \cap X' \neq \phi$. Suppose $(i, j') \in \Gamma(y) \cap X'$. Then $(i, j')^\sigma = (i, j'')$ for some $j'' \geqslant 2$, because $(i, 1)^\sigma = (i, 1)$ and $(i, j)^{\sigma'} = (i, j)$. We have proved that τ is in $\mathrm{Sym}(k-1) \times \mathrm{Sym}(k-1)$, or τ has the form

$$(i, j)^\tau = (i^{\tau_1}, j^{\tau_2}),$$

where τ_1 and τ_2 are permutations of $\{2, 3, \ldots, k\}$. Define

$$\begin{aligned} i^{\eta_1} &= 1 && \text{if} \quad i = 1, \\ &= i^{\tau_1} && \text{if} \quad 2 \leqslant i \leqslant k, \\ j^{\eta_2} &= 1 && \text{if} \quad j = 1, \\ &= j^{\tau_2} && \text{if} \quad 2 \leqslant j \leqslant k, \end{aligned}$$

and

$$(i, j)^\eta = (i^{\eta_1}, j^{\eta_2}).$$

Then η is an automorphism of Γ, which is an extension of σ. This completes the proof of Proposition 3.

It is obvious that if Γ is KC- homogeneous then $\langle \Gamma(x) \rangle$ is K-homogeneous for $x \in V\Gamma$. Moreover, $|\Gamma(x)| = |\Gamma(y)|$ for any two vertices $x, y \in V\Gamma$ by KC-homogeneity. Suppose $\{x, y\}$ is an edge. Then

$$\langle \Gamma(x) \rangle (y) = \Gamma(x) \cap \Gamma(y) = \langle \Gamma(y) \rangle (x).$$

Therefore, $\langle \Gamma(x) \rangle$ and $\langle \Gamma(y) \rangle$ have the same valency. On the other hand, K-homogeneous graphs having the same number of vertices and the same valency are isomorphic by Theorem 1. Hence, $\langle \Gamma(x) \rangle \cong \langle \Gamma(y) \rangle$ for any edge $\{x, y\}$. If Γ is connected, $\langle \Gamma(x) \rangle \cong \langle \Gamma(y) \rangle$ for any two vertices $x, y \in V\Gamma$. The proof of Theorem 2 is divided into several propositions.

PROPOSITION 4. *Let Γ be a connected KC-homogeneous graph. If $\langle \Gamma(x) \rangle \cong k \cdot K_1$ $(k \geqslant 2)$, that is, if $\langle \Gamma(x) \rangle$ contains k vertices and no edges, then either*

(a) $k = 2$ *and Γ is isomorphic to a circuit graph,*

(b) $k = 3$, $d(\Gamma) = 2$ *and Γ is isomorphic to Petersen's graph,*

(c) $d(\Gamma) = 2$ *and Γ is isomorphic to a complete bipartite graph $K_{k,k}$,*

(d) $d(\Gamma) = 3$ *and Γ is isomorphic to $L(K_{2,k+1})^c$, or*

(e) $k = 5$, $d(\Gamma) = 2$ *and Γ is isomorphic to $\square_5$.*

Proof. Let $(x = x_0, x_1, ..., x_{g-1})$ be any shortest circuit in Γ, where g is the girth of Γ. If $k = 2$, Γ is a circuit graph. In the following we shall assume that $k \geqslant 3$, and choose $y \in \Gamma(x) - \{x_1, x_{g-1}\}$. Suppose $g \geqslant 7$. Applying KC-homogeneity to $X = \{x_{g-1}, x, x_1, ..., x_{g-3}\}$, $X' = \{y, x, x_1, ..., x_{g-3}\}$ and $Y = \{x_{g-1}, x_{g-3}\}$, we get an isomorphism σ of $\langle X \rangle$ onto $\langle X' \rangle$ such that $|\Gamma(Y)| = |\Gamma(Y^\sigma)|$. On the other hand, for any isomorphism σ of $\langle X \rangle$ onto $\langle X' \rangle$, $Y^\sigma = \{y, x_{g-3}\}$. This is a contradiction, because $\Gamma(Y) = \{x_{g-2}\}$ and $\Gamma(Y^\sigma) = \phi$. Next, suppose $g = 6$. Since $\Gamma(x_5) \cap \Gamma(x_3) = \{x_4\}$ and $\langle \{x_5, x, x_1, x_2, x_3\} \rangle \cong \langle \{y, x, x_1, x_2, x_3\} \rangle$, we know that $|\Gamma(y) \cap \Gamma(x_3)| = 1$ by KC-homogeneity. Let $\Gamma(y) \cap \Gamma(x_3) = \{z\}$, $z' \in \Gamma(y) - \{x, z\}$, $X = \{x, x_1, x_2, x_5, x_4, y, z\}$, $X' = \{x, x_1, x_2, x_5, x_4, y, z'\}$ and $Y = \{x_2, x_4, z\}$. Then there is an isomorphism σ of $\langle X \rangle$ onto $\langle X' \rangle$ such that $|\Gamma(Y)| = |\Gamma(Y^\sigma)|$. But, this is a contradiction, because $Y^\sigma = \{x_2, x_4, z'\}$, $\Gamma(Y^\sigma) = \phi$ and $\Gamma(Y) = \{x_3\}$. Suppose $g = 5$ and $k \geqslant 4$. Then we obtain $|\Gamma(y) \cap \Gamma(x_2)| = 1$ as above. Let $\Gamma(y) \cap \Gamma(x_2) = \{z\}$ and $w \in \Gamma(x) - \{x_1, x_4, y\}$. Then we may derive a contradiction for $X = \{x, x_1, x_4, x_3, y, z\}$, $X' = \{x, w, x_4, x_3, y, z\}$ and $Y = \{x_1, x_3, z\}$. We have proved that $k = 3$ if $g = 5$. In this case, it is easily verified that Γ is isomorphic to Petersen's graph. Finally, suppose $g = 4$ and let $\mu = |\Gamma(x) \cap \Gamma(x_2)|$. Then we have $|\Gamma(x) \cap \Gamma(z)| = \mu$ for any $z \in \Gamma_2(x)$ by KC-homogeneity. Let Z be a μ-subset of $\Gamma(x)$. Then, we know that there is an element z in $\Gamma_2(x)$ such that $\Gamma(x) \cap \Gamma(z) = Z$ by applying KC-homogeneity to $X = \{x\} \cup (\Gamma(x) \cap \Gamma(x_2))$, $X' = \{x\} \cup Z$ and $Y = \Gamma(x) \cap \Gamma(x_2)$. This implies that

$$|\Gamma_2(x)| = \frac{k(k-1)}{\mu} \geqslant \binom{k}{\mu}.$$

Therefore, $\mu = 2$, $k-1$, or k. If $\mu = k$, then Γ is isomorphic to a complete bipartite graph $K_{k,k}$. Suppose $\mu = k-1$. The elements in $\Gamma(x)$ and $\Gamma_2(x)$ may be labeled as

$$\Gamma(x) = \{1, 2, \dots, k\},$$
$$\Gamma_2(x) = \{1', 2', \dots, k'\}$$

and i and j' are adjacent if $i \neq j$. Then $|\Gamma_3(x)| = 1$, and the structure of Γ is uniquely determined. It is easily verified that Γ is isomorphic to $L(K_{2,k+1})^c$. Suppose $\mu = 2 < k-1$ and let $\Gamma(y) \cap \Gamma(x_1) = \{x, y_1\}$ and $\Gamma(y_1) \cap \Gamma(x_2) = \{x_1, z\}$. Suppose $z \in \Gamma_2(x)$. Then for each 2-subset W of $\Gamma(x) - \{x_1, x_3\}$, we have $|\Gamma(x_2) \cap \Gamma(W)| = 1$ by applying KC-homogeneity to $X = \{x, x_1, x_2\} \cup W$, $X' = \{x, x_1, x_2\} \cup (\Gamma(x) \cap \Gamma(z))$ and $Y = \{x_2\} \cup W$. Thus $\binom{k-2}{2} = k-2$, so $k = 5$ and it is easily verified that $d(\Gamma) = 2$ and that Γ is isomorphic to $\square_5$. Suppose $z \in \Gamma_3(x)$. Since we have assumed that $k \geqslant 4$, we may choose $y' \in \Gamma(x) - \{x_1, x_3, y\}$. We may derive a contradiction by applying KC-homogeneity to $X = \{y, x, x_1, x_2, z\}$, $X' = \{y', x, x_1, x_2, z\}$ and $Y = \{y, x_1, z\}$.

PROPOSITION 5. *Let Γ be a connected KC-homogeneous graph. If $\langle \Gamma(x) \rangle$ is isomorphic to $t \cdot K_r$ $(t, r \geqslant 2)$, then $t = 2$ and Γ is isomorphic to $L(K_{r+1,r+1})$.*

Proof. A sequence $(x_0, x_1, \dots, x_{g'-1})$ of vertices is called an irreducible cycle of length g' if $g' \geqslant 4$ and

$$x_i \in \Gamma(x_j) \qquad \text{if and only if} \quad i \equiv j \pm 1 \bmod g'.$$

Let $C = (x, x_1, \dots, x_{g'-1})$ be an irreducible cycle of minimal length. Choose $x' \in \Gamma(x) \cap \Gamma(x_1)$. Then $x_i \notin \Gamma(x')$ for $i \neq 1$. Indeed, $\langle \Gamma(x') \cap \Gamma(x_1) \rangle$ is a complete subgraph, because $\langle \Gamma(x') \rangle \cong t \cdot K_r$. Since $x \in \Gamma(x') \cap \Gamma(x_1)$ and x_2 is not adjacent to x, we conclude that $x_2 \notin \Gamma(x') \cap \Gamma(x_1)$. Similarly, $x_{g'-1} \notin \Gamma(x')$. Then the minimality of the irreducible cycle C implies that $x_i \notin \Gamma(x')$ for $3 \leqslant i \leqslant g'-2$. If $g' \geqslant 6$, we may derive a contradiction for $X = \{x_3, x_4, \dots, x_{g'-1}, x, x_1\}$, $X' = \{x_3, x_4, \dots, x_{g'-1}, x, x'\}$ and $Y = \{x_3, x_1\}$. Suppose $g' = 5$ and $t \geqslant 3$. Let $y \in \Gamma(x) \cap \Gamma_2(x_1) \cap \Gamma_2(x_4)$. Then $|\Gamma(y) \cap \Gamma(x_2)| = 1$ by KC-homogeneity, and let $\Gamma(y) \cap \Gamma(x_2) = \{z\}$. By applying KC-homogeneity to $X = \{x, x_1, x_4, x_3, y, z\}$, $X' = \{x, x', x_4, x_3, y, z\}$ and $Y = \{x_1, x_3, z\}$, we get a contradiction. Suppose $g' = 5$ and $t = 2$. We know $|\Gamma(x') \cap \Gamma(x_3)| = 1$ by KC-homogeneity. Let $\Gamma(x') \cap \Gamma(x_3) = \{z\}$. Then $z \in \Gamma(x_2) \cup \Gamma(x_4)$, because $t = 2$. But this is impossible. Finally, suppose $g' = 4$ and let $\mu = |\Gamma(x) \cap \Gamma(x_2)|$. Then for every edge-free μ-subset Z in $\Gamma(x)$, there is a vertex z in $\Gamma_2(x)$ such that $\Gamma(x) \cap \Gamma(z) = Z$. Therefore

$$|\Gamma_2(x)| = \frac{tr(tr-r)}{\mu} \geqslant \binom{t}{\mu} r^{\mu}.$$

This implies that $\mu = 2$ or t. Suppose $\mu = t$. Then $r = 2$ and $t = 3$. This case is missing in [2–4], but it is not difficult to prove that a strongly regular graph with these parameters is isomorphic to $L(K_6)^c$, which is not KC-homogeneous. Suppose $\mu = 2$. If $t \geqslant 3$, choose $y \in \Gamma(x) \cap \Gamma_2(x_1) \cap \Gamma_2(x_3)$. Then applying KC-homogeneity to $X = \{x', x, x_3, x_2\}$, $X' = \{y, x, x_3, x_2\}$ and $Y = \{x', x, x_2\}$, we get an isomorphism σ of $\langle X \rangle$ onto $\langle X' \rangle$ such that $|\Gamma(Y^\sigma)| = |\Gamma(Y)| = 1$. The only possibility for Y^σ is $\{y, x_3, x_2\}$, because $\Gamma(\{y, x, x_2\}) = \phi$. This implies that $\Gamma(y) \cap \Gamma(x_3) \cap \Gamma(x_2) \neq \phi$. But this yields a contradiction since there are at least r candidates for y and only $r-1$ vertices in $\Gamma(x_3) \cap \Gamma(x_2)$. If $t = 2$, we have

$$|\{x_1, x_3\} \cup (\Gamma(x_2) \cap \Gamma(x_1)) \cup (\Gamma(x_2) \cap \Gamma(x_3))| = 2r = |\Gamma(x)|,$$

which shows that $d(\Gamma) = 2$. Then it is easily verified that $\Gamma \cong L(K_{r+1,r+1})$, and we have completed the proof of Proposition 5.

PROPOSITION 6 [3, Lemmas 6 and 8]. *Let Γ be a connected graph with $\langle \Gamma(x) \rangle \cong K_{t;r}$ $(t \geqslant 2,\ r \geqslant 1)$ for every $x \in V\Gamma$. Then $\Gamma \cong K_{t+1;r}$.*

PROPOSITION 7 [3, Lemma 9(iii)]. *Let Γ be a connected graph with $\langle \Gamma(x) \rangle \cong C_5$ for every $x \in V\Gamma$. Then Γ is isomorphic to the icosahedron and is not KC-homogeneous.*

PROPOSITION 8. *Let Γ be a KC-homogeneous graph. Then $\langle \Gamma(x) \rangle$ is not isomorphic to $L(K_{3,3})$ for $x \in V\Gamma$.*

Proof. Suppose $\Gamma(x)$ is isomorphic to $L(K_{3,3})$. Then we may identify $\Gamma(x)$ with the set of all ordered pairs of three symbols, that is,

$$\Gamma(x) = \{(i, j) \mid 1 \leqslant i, j \leqslant 3\},$$

and (i, j) and (k, m) are adjacent if $i = k$ or $j = m$. Let $y_1 = (1, 1)$ and $y_2 = (2, 2)$. Then we have

$$\Gamma(y_1) \cap \Gamma(y_2) \cap \Gamma(x) = \{(1, 2), (2, 1)\}.$$

This implies that $\langle \Gamma(y_1) \cap \Gamma(y_2) \rangle$ is regular of valency 2. Therefore, $\langle \Gamma(x) \cap \Gamma(z) \rangle$ is a union of circuits of length $\geqslant 4$ whenever $z \in \Gamma_2(x)$. There are two possibilities: a quadrangle or a hexagon. If $\langle \Gamma(x) \cap \Gamma(z) \rangle$ is isomorphic to a quadrangle, then $\Gamma_2(x) \cong L(K_{3,3})$ and we may parametrize $\Gamma_2(x)$ as

$$\Gamma_2(x) = \{(i', j') \mid 1 \leqslant i, j \leqslant 3\},$$

and (i, j) and (k', m') are adjacent if $i \neq k$ and $j \neq m$; (i', j') and (k', m') are adjacent if $i = k$ or $j = m$. Then it is easily proved that $|\Gamma_3(x)| = 1$. Let $\Gamma_3(x) = \{w\}$. Then $\Gamma(w) = \Gamma_2(x)$ and we may derive a contradiction by applying KC-homogeneity to $X = \{x, (1, 1), (2', 2'), (1', 2')\}$, $X' = \{x, (1, 1), (2', 2'), w\}$ and $Y = \{x, (1', 2')\}$. If $\langle \Gamma(x) \cap \Gamma(z) \rangle$ is isomorphic to a hexagon, then we may identify $\Gamma_2(x)$ with the set of permutations on three letters,

$$\Gamma_2(x) = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ i_1 & i_2 & i_3 \end{pmatrix} \right\},$$

and $\left(\begin{smallmatrix} 1 & 2 & 3 \\ i_1 & i_2 & i_3 \end{smallmatrix}\right)$ is non-adjacent to $(k, i_k)(1 \leqslant k \leqslant 3)$ and adjacent to all the other vertices of $\Gamma(x)$. Then $\left(\begin{smallmatrix} 1 & 2 & 3 \\ i_1 & i_2 & i_3 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 2 & 3 \\ j_1 & j_2 & j_3 \end{smallmatrix}\right)$ are adjacent if $i_k = j_k$ for some k. We have proved that Γ is isomorphic to $L(K_{4,4})^c$, and it is easily verified that $L(K_{4,4})^c$ is not KC-homogeneous. This completes the proof of Proposition 8 and Theorem 2.

Remark 1. $L(K_{4,4})^c$ was missed in the original manuscript (and also in [4, Lemma 10]). The author is grateful to the referee, who pointed out the error.

Remark 2. A direct proof that combinatorially C-homogeneous graphs are C-ultrahomogeneous has not yet been obtained. Possibly, our definition of KC-homogeneity presented in this paper might be inadequate for this purpose.

Remark 3. In this paper, we have restricted ourselves to finite graphs only, but it is not difficult to classify locally finite KC-homogeneous graphs by modifying the proof of [4, Theorem 3], though there is a gap in the proof of [4, Lemma 6]. The graph T_t^r defined below should be added to the list of locally finite C-ultrahomogeneous graphs,

$$VT_t^r = \left\{ \begin{pmatrix} a_1 & \cdots & a_m \\ b_1 & \cdots & b_m \end{pmatrix} \;\middle|\; m \geqslant 0,\ 1 \leqslant a_i \leqslant r,\ 1 \leqslant b_i \leqslant t,\ b_i \neq b_{i+1} \right\}.$$

and $\left(\begin{smallmatrix} a_1 \cdots a_m \\ b_1 \cdots b_m \end{smallmatrix}\right)$ is adjacent to $\left(\begin{smallmatrix} a_1 \cdots a_{m-1} \\ b_1 \cdots b_{m-1} \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} a_1 \cdots a_{m-1} a'_m \\ b_1 \cdots b_{m-1} b_m \end{smallmatrix}\right)$ $(a'_m \neq a_m)$ and $\left(\begin{smallmatrix} a_1 \cdots a_m a_{m+1} \\ b_1 \cdots b_m b_{m+1} \end{smallmatrix}\right)$. Note that $T_t^1 \cong T_t$ (infinite regular tree with valency t), $T_2^{t-1} \cong L(T_t)$ and $\langle T_t^r(x) \rangle \cong t \cdot K_r$ for $x \in VT_t^r$.

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